22 Solving linear systems of ODE with constant coefficients. Part III. Eigenvalues with multiplicities two and more. Matrix exponent

To deal with the eigenvalues of multiplicity 2 and above is mathematically challenging in the general form, therefore I divide this lecture into two parts. In the first one I somewhat heuristically deal with the case of a two by two matrix $\boldsymbol{A}$ that has an eigenvalue λ of (algebraic) multiplicity two and consider two possible cases. In the second part, which in principle covers any situation, I introduce the so-called matrix exponent. The second part of this lecture is usually optional for an introductory ODE class.

22.1 The case of a two by two matrix

Assume that we must solve a system od first order ODE of the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(t) \in \mathbf{R}^2,$$

and it happened that matrix $\boldsymbol{A}$ has only one eigenvalue λ of (algebraic) multiplicity two. For the corresponding eigenvector(s) it is possible in this case to have two quite different situations. First, it is feasible to have two linearly independent eigenvectors correspond to the same eigenvalue λ. As an example take, e.g., matrix $\boldsymbol{I}_2$, which clearly has only one eigenvalue $\lambda = 1$ of (algebraic) multiplicity 2. Solving the system

$$(\boldsymbol{I}_2 - \lambda \boldsymbol{I}_2)\boldsymbol{v} = 0$$

however yields that *any* vector $\boldsymbol{v} \in \mathbf{R}^2$ is a solution, and to characterize all the solutions we can take any two linearly independent vectors, which form a basis of $\mathbf{R}^2$. I can, e.g., take $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as a basis. In this case the general solution to the corresponding system of ODE is written in general as

$$\boldsymbol{y}(t) = C_1 \boldsymbol{v}_1 e^{\lambda t} + C_2 \boldsymbol{v}_2 e^{\lambda t},$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2$ ate two linearly independent eigenvectors corresponding to λ (it is said that λ has the geometric multiplicity two as well in this case). How often it can happen? Not very often. It can be proved (an exercise for a mathematically inclined student) that a two by two matrix $\boldsymbol{A}$ has one eigenvalue λ of (algebraic) multiplicity 2 with two linearly independent eigenvectors if and only if $\boldsymbol{A} = a\boldsymbol{I}_2$, i.e, $\boldsymbol{A}$ is a scalar product of diagonal matrix. Not a very interesting case.

In most cases one finds that if $\boldsymbol{A}$ has one eigenvalue λ with multiplicity two, there is only one linearly independent eigenvector. That is, we know one solution to the ODE system, but lacking the second one. Let me try, using our experience with linear ODE with constant coefficients, try to look for a solution in the form

$$\boldsymbol{y}(t) = \boldsymbol{v} t e^{\lambda t} + \boldsymbol{w} e^{\lambda t}.$$

Plugging this expression into my equation I find

$$\boldsymbol{v} e^{\lambda t} + \lambda \boldsymbol{v} t e^{\lambda t} + \lambda \boldsymbol{w} e^{\lambda t} = \boldsymbol{A}\boldsymbol{v} t e^{\lambda t} + \boldsymbol{A}\boldsymbol{w} e^{\lambda t}.$$

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

After cancelling the exponent and comparing the coefficients at the same powers of t I get

$$\begin{aligned}\boldsymbol{A}\boldsymbol{v} &= \lambda\boldsymbol{v},\\ \boldsymbol{A}\boldsymbol{w} &= \lambda\boldsymbol{w}+\boldsymbol{v},\end{aligned}$$

which gives me the recipe to find the second solution. The first equation is simply the eigenvector problem for $\boldsymbol{v}$, and hence the second equation is a system of linear equation with respect to unknown $\boldsymbol{w}$, which (it requires proof!) can be always solved. Here is an example.

Example 1. Solve

$$\dot{\boldsymbol{y}} = \begin{bmatrix}3 & -18\\ 2 & -9\end{bmatrix}\boldsymbol{y}.$$

The characteristic equation is

$$\lambda^2+6\lambda+9=(\lambda+3)^2=0,$$

hance there is one eigenvalue -3 multiplicity two. By solving the corresponding homogeneous system I find that there is only one linearly independent vector $\boldsymbol{v}=(3,1)^\top$, and hence one linearly independent solution is

$$\boldsymbol{y}_1(t)=\begin{bmatrix}3\\1\end{bmatrix}e^{-3t}.$$

Now to find $\boldsymbol{w}=(w_1,w_2)^\top$ I need to solve

$$\begin{bmatrix}6 & -18\\ 2 & -6\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix},$$

which has infinitely many solutions. I need only one, hence from the first equation

$$w_1=3w_2+1/2$$

I can take

$$\boldsymbol{w}=\begin{bmatrix}1/2\\0\end{bmatrix},$$

and therefore my second solution is

$$\boldsymbol{y}_2(t)=\begin{bmatrix}3\\1\end{bmatrix}te^{-3t}+\begin{bmatrix}1/2\\0\end{bmatrix}e^{-3t},$$

and the general solution to my problem is given by

$$\boldsymbol{y}(t)=C_1\boldsymbol{y}_1(t)+C_2\boldsymbol{y}_2(t).$$

Remark 2. To connect this subsection with the following I note that the equation

$$\boldsymbol{A}\boldsymbol{w}=\lambda\boldsymbol{w}+\boldsymbol{v}$$

implies (why?) that

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{w}=0,$$

which sometimes can be used to determine the unknown $\boldsymbol{w}$.

22.2 *Matrix exponent

Since "guessing" a solution is not very mathematically attractive, here I present a rigorous and general approach to deal with repeated eigenvalues. It is based on the mathematical object, which is called *matrix exponent.*

Consider the following first order differential equation of the form

$$y' = ay, \quad a \in \mathbf{R},$$

with the initial condition

$$y(0) = y_0.$$

Of course, we know that the solution to this IVP is given by

$$y(t) = e^{at} y_0.$$

However, let us apply the method of iterations to this equation. First note that instead of differential equation plus the initial conditions we can have one *integral equation*

$$y(t) = y_0 + \int_0^t ay(\tau)\,\mathrm{d}\tau.$$

Now we plug in the right hand side $y(\tau) = y_0$ and find first iteration $y_1(t)$:

$$y_1(t) = y_0 + \int_0^t ay_0\,\mathrm{d}\tau = y_0 + ay_0 t = (1 + at)y_0.$$

I plug it again in the right hand side and find the second iteration

$$y_2(t) = y_0 + \int_0^t ay_1(\tau)\,\mathrm{d}\tau = \left(1 + at + \frac{a^2t^2}{2}\right) y_0.$$

In general we find

$$y_n(t) = y_0 + \int_0^t ay_{n-1}(\tau)\,\mathrm{d}\tau = \left(1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \ldots + \frac{a^nt^n}{n!}\right) y_0.$$

You should recognize inside the parenthesis the partial sums for the Taylor series of e^{at}, hence we recover again our familiar solution

$$y(t) = e^{at} y_0.$$

So what is the point about these iterations? Let us do the same trick with the system

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{1}$$

Instead of (1) we can write the integral equation

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_0^t \boldsymbol{A}\boldsymbol{y}(\tau)\,\mathrm{d}\tau,$$

where the integral of a vector is understood as componentwise integrals. I plug in the right-hand side $\boldsymbol{y}_0$ and find the first iteration

$$\boldsymbol{y}_1(t) = \boldsymbol{y}_0 + t\boldsymbol{A}\boldsymbol{y}_0 = (\boldsymbol{I} + \boldsymbol{A}t)\boldsymbol{y}_0.$$

Similarly to the previous, we find

$$\boldsymbol{y}_n(t) = \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \ldots + \frac{\boldsymbol{A}^nt^n}{n!}\right)\boldsymbol{y}_0.$$

The expression in the parenthesis is a sum of $n \times n$ matrices, and hence a matrix itself. Therefore, it is natural to *define* a matrix, which is called *matrix exponent*, as the infinite sum of the form:

$$e^{\boldsymbol{A}t} = \exp \boldsymbol{A}t := \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2t^2}{2!} + \ldots + \frac{\boldsymbol{A}^nt^n}{n!} + \ldots$$

Note that we can include scalar t to the matrix $\boldsymbol{A}$.

Definition 3. *The matrix exponent $e^{\boldsymbol{A}}$ of $\boldsymbol{A}$ is the series*

$$e^{\boldsymbol{A}} = \exp \boldsymbol{A} := \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \ldots + \frac{\boldsymbol{A}^n}{n!} + \ldots \tag{2}$$

To make sure that the definition makes sense we need to specify what we understand under the infinite series of matrices. I will skip this point here and just mention that series (2) converges absolutely for any matrix $\boldsymbol{A}$, which allows us multiply this series by another matrix, differentiate it term by term, or integrate it if necessary.

Matrix exponent has a lot of properties similar to the usual exponent. Here are those that I will need in the following:

1. As I already mentioned, series (2) converges absolutely, which means that there is a well defined limit of partial sums of this series.

2. $$\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}.$$

   This property can be proved by term by term differentiation and factoring out $\boldsymbol{A}$ (left as an exercise). Note here that both $\boldsymbol{A}$ and $e^{\boldsymbol{A}t}$ are $n \times n$ matrices, and it is not obvious that $\boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$. Such matrices for which $\boldsymbol{AB} = \boldsymbol{BA}$ are called *commuting.*

3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, then
   $$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}}.$$

   In particular $\boldsymbol{A}$ and $\boldsymbol{B}$ commute if one of them is a scalar matrix, i.e., it has the form of the form $\lambda\boldsymbol{I}$.

4. $$e^{\lambda\boldsymbol{I}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v},$$

   for any $\lambda \in \mathbf{R}$ and $\boldsymbol{v} \in \mathbf{R}^n$. The proof follows from the definition.

Before using the matrix exponent to solve problems with equal eigenvalues, I would like to state the fundamental theorem of linear first order homogeneous ODE with constant coefficients:

Theorem 4. *Consider problem* (1). *Then this problem has the unique solution*

$$\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0.$$

Moreover, for any vector $\boldsymbol{v} \in \mathbf{R}^n$, $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{v}$ *is a solution to the system* $\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}$.

22.3 Dealing with equal eigenvalues

It is important to note that the matrix exponent is not that easy to calculate for each particular example. However, the expression $e^{\boldsymbol{A}t}\boldsymbol{v}$ can be easily calculated for some special vectors $\boldsymbol{v}$ without the knowledge of the explicit form of $e^{\boldsymbol{A}t}$.

Example 5. For eigenvector $\boldsymbol{v}$ with the eigenvalue λ we have that

$$e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}.$$

To show this, express $\boldsymbol{A}t = \lambda\boldsymbol{I}t + \boldsymbol{A}t - \lambda\boldsymbol{I}t$, then

$$\begin{aligned}
e^{\boldsymbol{A}t}\boldsymbol{v} &= e^{\lambda\boldsymbol{I}t+\boldsymbol{A}t-\lambda\boldsymbol{I}t}\boldsymbol{v} = \text{ by property 3} \\
&= e^{\lambda\boldsymbol{I}t}e^{(\boldsymbol{A}-\lambda\boldsymbol{I})t}\boldsymbol{v} = \text{ by property 4 } = e^{\lambda t}e^{(\boldsymbol{A}-\lambda\boldsymbol{I})t}\boldsymbol{v} = \text{ by definition} \\
&= e^{\lambda t}\left(\boldsymbol{I} + (\boldsymbol{A}-\lambda\boldsymbol{I})t + \frac{(\boldsymbol{A}-\lambda\boldsymbol{I})^2t^2}{2!} + \ldots\right)\boldsymbol{v} \\
&= e^{\lambda t}\left(\boldsymbol{I}\boldsymbol{v} + t(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v} + \frac{t^2(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v}}{2!} + \ldots\right) = \text{ by the properties of the eigenvectors} \\
&= e^{\lambda t}(\boldsymbol{I}\boldsymbol{v} + 0 + 0 + \ldots) = e^{\lambda t}\boldsymbol{v}.
\end{aligned}$$

We actually found exactly those solutions to system $\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}$ that can be written down using the distinct eigenvalues.

Definition 6. *A nonzero vector* $\boldsymbol{v}$ *is called a generalized eigenvector of matrix* $\boldsymbol{A}$ *associated with the eigenvalue* λ *with the algebraic multiplicity* $k > 1$, *if*

$$(\boldsymbol{A} - \lambda\boldsymbol{I})^k\boldsymbol{v} = 0.$$

Now assume that vector $\boldsymbol{v}$ is a generalized eigenvector with $k = 2$. Exactly as in the last example, we will find that

$$\begin{aligned}
e^{\boldsymbol{A}t}\boldsymbol{v} &= e^{\lambda\boldsymbol{I}t+\boldsymbol{A}t-\lambda\boldsymbol{I}t}\boldsymbol{v} = \text{ by property 3} \\
&= e^{\lambda\boldsymbol{I}t}e^{(\boldsymbol{A}-\lambda\boldsymbol{I})t}\boldsymbol{v} = \text{ by property 4 } = e^{\lambda t}e^{t(\boldsymbol{A}-\lambda\boldsymbol{I})}\boldsymbol{v} = \text{ by definition} \\
&= e^{\lambda t}\left(\boldsymbol{I} + t(\boldsymbol{A}-\lambda\boldsymbol{I}) + \frac{t^2(\boldsymbol{A}-\lambda\boldsymbol{I})^2}{2!} + \frac{t^3(\boldsymbol{A}-\lambda\boldsymbol{I})^3}{3!} + \ldots\right)\boldsymbol{v} \\
&= e^{\lambda t}\left(\boldsymbol{I}\boldsymbol{v} + t(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v} + \frac{t^2(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v}}{2!} + \frac{t^3(\boldsymbol{A}-\lambda\boldsymbol{I})^3\boldsymbol{v}}{3!} + \ldots\right) = \text{ by the properties of the eigenvectors} \\
&= e^{\lambda t}\big(\boldsymbol{I}\boldsymbol{v} + t(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v} + 0 + 0 + \ldots\big) = e^{\lambda t}\big(\boldsymbol{I} + t(\boldsymbol{A}-\lambda\boldsymbol{I})\big)\boldsymbol{v}.
\end{aligned}$$

Hence we found that for a generalized eigenvector $\boldsymbol{v}$ with $k=2$, the solution to our system can be taken as

$$\boldsymbol{y}(t)=e^{\lambda t}\big(\boldsymbol{I}+t(\boldsymbol{A}-\lambda\boldsymbol{I})\big)\boldsymbol{v},$$

which does not require much computations. The only remaining question is actually whether we are always able to find enough linearly independent generalized eigenvectors for a given matrix. The answer is positive. Hence we obtain an algorithm for matrices with equal eigenvalues:

- Assume that we have a real eigenvalue λ_i of multiplicity 2 and we found only one linearly independent eigenvector $\boldsymbol{v}_i$ corresponding to this eigenvalue (if we are able to find two, the problem is solved). Then first particular solution is given by, as before,

  $$\boldsymbol{y}_i(t)=\boldsymbol{v}_i e^{\lambda_i t}.$$

  To find a second particular solution to account for this multiplicity we need to look for a generalized eigenvector that solves the equation

  $$(\boldsymbol{A}-\lambda_i\boldsymbol{I})^2\boldsymbol{u}_i=\boldsymbol{0}.$$

  Note that we are looking for such $\boldsymbol{u}_i$ that the previous holds *and*

  $$(\boldsymbol{A}-\lambda_i\boldsymbol{I})\boldsymbol{u}_i\neq\boldsymbol{0}.$$

  We can always find a solution $\boldsymbol{u}_i$ of this system, which is linearly independent of $\boldsymbol{v}_i$. In this case the second particular solution is given by

  $$\boldsymbol{y}_{i+1}(t)=e^{\lambda_i t}\big(\boldsymbol{I}+(\boldsymbol{A}-\lambda_i\boldsymbol{I})t\big)\boldsymbol{u}_i.$$

  This case can be generalized to the case when multiplicity of eigenvalues in bigger than 2 (see an example below) and when we have complex conjugate eigenvalues of multiplicity two and higher (we will not need this case for the quizzes and exams).

Example 7. Find the general solution to

$$\dot{\boldsymbol{y}}=\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}\boldsymbol{y}.$$

The eigenvalues are $\lambda_1=2$ and $\lambda_2=1$ (multiplicity 2). An eigenvector for λ_1 can be taken as

$$\boldsymbol{v}_1=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

For λ_2 we find that

$$\boldsymbol{v}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},$$

and we are short for one more linearly independent solution to form a basis for the solution set. Consider

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})^2 \boldsymbol{u} = \boldsymbol{0},$$

which has linearly independent solutions

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The first one is exactly $\boldsymbol{v}_2$, therefore we keep only $\boldsymbol{u}_2$.

Finally, one finds that

$$\boldsymbol{y}_3(t) = e^t \big(\boldsymbol{I} + (\boldsymbol{A} - \lambda_2 \boldsymbol{I})t\big)\boldsymbol{u}_2 = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} e^t,$$

and the general solution is

$$\boldsymbol{y}(t) = C_1 \boldsymbol{v}_1 e^{2t} + C_2 \boldsymbol{v}_2 e^t + C_3 \boldsymbol{y}_3(t).$$

Example 8. Solve the IVP

$$\boldsymbol{y} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

I find that $\lambda = 2$ is the only eigenvalue of multiplicity 3. Its eigenvector is

$$\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

and a first linearly independent solution is given by

$$\boldsymbol{y}_1(t) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t}.$$

To find two more linearly independent solutions we need to look for the generalized eigenvectors. Consider first

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})^2 \boldsymbol{u} = \boldsymbol{0},$$

which has two vectors as a solution basis

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Note that $\boldsymbol{v}, \boldsymbol{u}_1, \boldsymbol{u}_2$ are linearly dependent (why?) and therefore we can keep only one of the vectors, e.g., $\boldsymbol{u}_1$. The particular solution in this case

$$\boldsymbol{y}_2(t) = e^{2t}\big(\boldsymbol{I} + (\boldsymbol{A} - 2\boldsymbol{I})t\big)\boldsymbol{u}_1 = \begin{bmatrix} 1-t \\ t \\ t \end{bmatrix} e^{2t}.$$

To find one more linearly independent solution let us look for a generalized eigenvector with $k = 3$:

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})^3 \boldsymbol{w} = \boldsymbol{0}.$$

Note that $(\boldsymbol{A} - \lambda_2 \boldsymbol{I})^3 = 0$ therefore any vector $\boldsymbol{w}$ will do, the only thing is that we need it to be linearly independent of $\boldsymbol{v}$ and $\boldsymbol{u}_1$. For instance we can take

$$\boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the last solution is given by

$$\boldsymbol{y}_3(t) = e^{2t}\left(\boldsymbol{I} + (\boldsymbol{A} - 2\boldsymbol{I})t + \frac{1}{2}(\boldsymbol{A} - 2\boldsymbol{I})^2 t^2\right)\boldsymbol{w} = \begin{bmatrix} -3t + \frac{t^2}{2} \\ 2t - \frac{t^2}{2} \\ 1 + 2t - \frac{t^2}{2} \end{bmatrix} e^{2t},$$

and hence the general solution to our problem is

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1-t \\ t \\ t \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} -3t + \frac{t^2}{2} \\ 2t - \frac{t^2}{2} \\ 1 + 2t - \frac{t^2}{2} \end{bmatrix} e^{2t}.$$

Applying the initial conditions, one finds $C_1 = C_3 = 0$, $C_2 = 1$ and finally the solution is

$$\boldsymbol{y}(t) = \begin{bmatrix} 1-t \\ t \\ t \end{bmatrix} e^{2t}.$$

22.4 How to find $e^{\boldsymbol{A}t}$

Recall that if we are given system

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \tag{3}$$

then its solution space is an n-dimensional vector space and has a basis $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$. The matrix $\boldsymbol{\Phi}(t)$, which has $\boldsymbol{y}_i(t)$ as its i-th column is called the *fundamental matrix solution*:

$$\boldsymbol{\Phi}(t) = \big(\boldsymbol{y}_1(t)| \ldots |\boldsymbol{y}_n(t)\big).$$

Theorem 9. *Let $\boldsymbol{\Phi}(t)$ be a fundamental matrix solution of* (3)*, then*

$$e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0).$$

Proof. First note that if $\boldsymbol{\Phi}(t)$ is a fundamental matrix solution it solves the matrix differential equation

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{A}\boldsymbol{\Phi}.$$

Moreover, since the columns of $\boldsymbol{\Phi}(t)$ are linearly independent, then $\det \boldsymbol{\Phi}(0) \neq 0$. Now, since

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\boldsymbol{A}t} = \boldsymbol{A} e^{\boldsymbol{A}t}, \quad e^{\boldsymbol{A}0} = \boldsymbol{I},$$

then $e^{\boldsymbol{A}t}$ is a fundamental matrix solution itself. For any two fundamental matrix solutions $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ it is true that

$$\boldsymbol{X}(t) = \boldsymbol{Y}(t)\boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant matrix. The last equality is true since each column of $\boldsymbol{X}(t)$ can be expressed as a linear combination of columns of $\boldsymbol{Y}(t)$. Therefore, by plugging $t = 0$, we find

$$e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{C} \Longrightarrow \boldsymbol{C} = \boldsymbol{\Phi}^{-1}(0).$$

■

This approach is not usually the best to find $e^{\boldsymbol{A}t}$ and requires quite a few calculations.

Example 10. Consider system (3) with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

To find the fundamental matrix solution, we find eigenvalues and eigenvectors of $\boldsymbol{A}$:

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}.$$

Next,

$$\boldsymbol{\Phi}^{-1}(0) = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Finally,

$$e^{\boldsymbol{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0) = \begin{bmatrix} e^{t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t} + e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}.$$